(partial) Notes ~~of~~ of

Johan's Seminar on (Stable) Rationality.

Speaker: us.

Time: Spring 2016.

Rationality. all schemes X here are ——

Defn. $CH^*(X)$: $Z_i(X) = \bigoplus_{\substack{V \text{ is a} \\ \text{subvty of dim } i \\ \text{in } X}} \mathbb{Z} \cdot V$ $A_i(X) = Z$

If V_1, V_2 are k-dim'l subvties in X. TFAE (& Defn)

rat'l eq. ① $\exists$ W, $(k+1)$-dim'l $\subseteq X$, a rat'l function f on W. s.t. $V_1 - V_2 = \operatorname{div}(f)$.

② $V_1 \sim V_2$, $\exists$ W subvty of $X \times \mathbb{P}^1$ dominantly projection on $\mathbb{P}^1$ s.t. $W \cap (X \times \{0\}) = V_1$, $W \cap (X \times \{\infty\}) = V_2$

e.g. $A_i(\mathbb{A}^n) = 0$, with $i < n$

pf. $W^\circ = \{(z,t) \mid \frac{z}{t} \in Z\} \subseteq \mathbb{A}^n \times (\mathbb{A}^1 \setminus \{0\})$

take $W = \overline{W^\circ}$ in $\mathbb{A}^n \times \mathbb{P}^1$.

Defn When X is smooth, there is a product:

$A, B \subseteq X$, intersect generically transversely, if $\forall$ generic point of $A \cap B$, the following is true: A, B are smooth at P and the tangent spaces to them span $T_P X$.

$[A] \cdot [B] = [A \cap B]$ if they intersect generically transversely.

Moving lemma: X smooth + proj:

a) $\forall$ $\alpha, \beta \in CH^*(X)$, there are cycles $A = \sum m_i A_i$, $B = \sum n_j B_j$, s.t. $[A] = \alpha$, $[B] = \beta$, each pair of $A_i \cap B_j$ intersect generically transversely.

b) the class $\sum m_i n_j [A_i \cap B_j]$ depends only on α and β.

- CH^* has proper pushforward, if $X \xrightarrow{f} Y$ is proper, then $\exists\, f_*: CH^*(X) \longrightarrow CH^*(Y)$, take direct image w/ multiplicity the extn deg of function field (right dim'l comp.)

- If $X \xrightarrow{f} Y$ is flat, then $\exists\, CH^*(Y) \xrightarrow{f^*} CH^*(X)$, taking preimage of rel. dim. given, which is a ring hom. if X, Y are smooth.
If X and Y are smooth, $f: X \to Y$ proper, it's defined by moving Y intersect $f(X)$ gen. transversely.

Prop. $f_*(\alpha \cdot (f^*\beta)) = (f_*\alpha) \cdot \beta$, for $f: X \to Y$ proper between smooth.

(adjunction formula)

Prop (Localization formula) $Y \subseteq X$ closed subscheme, $U = X \setminus Y$, then
$CH_*(Y) \longrightarrow CH_*(X) \longrightarrow CH_*(U) \longrightarrow 0$ is exact

e.g. $CH^*(\mathbb{P}^n) = \mathbb{Z}[H]/(H^{n+1})$, H of deg 1.

Rat'ly connected variety:

Refn X proper and a generic pair of pts may be conn. by a rat'l curve

Prop. $CH_0(X) = \mathbb{Z}$ if X is rat'lly conn. and $k = \bar{k}$.
pf. Z - subvariety of pts not connected to x, and then moving lemma

Thm $CH_0(X)$ is k-stably birat'l invariant for smooth proj. X.
i.e., $X \times \mathbb{P}^n \dashrightarrow Y \times \mathbb{P}^m$, then $CH_0(X) = CH_0(Y)$.
pf. ① $CH_0(X \times \mathbb{P}^n) = CH_0(X)$ by localization formula
② CH_0 is birat'l invariant, graph gives correspondence

$$
\begin{tikzcd}
 & X\times Y\times X \arrow[d] \arrow[dl] \arrow[dr] & \\
 & X\times X \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
X\times Y \arrow[d] \arrow[dr] & & Y\times X \arrow[dl] \arrow[d] \\
X & Y & X
\end{tikzcd}
$$

and the composition of two graphs as correspondence has decomposition $= \Delta_X + \cancel{Z \times Z_1}\,[D]$, where
$D \subseteq Z_1 \times Z_1$, and $X \setminus Z_1 \simeq Y \setminus Z_2$.
so we are done ~~by~~.

X smooth proj $/k$. (we may replace rat'l pt to a 0-cycle of deg 1 everywhere)

Defn X admits a decomposition of the diagonal if in $CH_n(X\times X)$

$$\Delta_X = X\times\{x\} + Z, \text{ where } Z\subseteq D\times X \text{ w/ } D\subseteq X \text{ proper closed}, x\in X(k).$$

coh. d.d. if $[\Delta_X] = [X\times\{x\}]+[Z]$ in $H^{2n}(X\times X, \mathbb{Z})$.

e.g. $X=\mathbb{P}^n$ $CH^*(\mathbb{P}^n\times\mathbb{P}^n) = \mathbb{Z}[\alpha,\beta]/(\alpha^{n+1},\beta^{n+1})$

Claim: $\Delta_{\mathbb{P}^n} = \alpha^n + \alpha^{n-1}\beta + \cdots + \beta^n = \beta^n + \alpha(\Delta_H)$ (d.d. holds).

Lemma. If X is d.d. for $x\in X(k)$, then $CH_0(X)\xrightarrow{\sim}\mathbb{Z}$

pf. $\Delta_X = X\times\{x\}+Z$, fix $z\in CH_0(X)$

$\Delta_X\cdot z = z$, while $(X\times\{x\}+Z)\cdot z = \deg(z)\cdot x$.

Partial converse If $X/\mathbb{C}$, $CH_0(X)=\mathbb{Z}$, then X has d.d. w/ $\mathbb{Q}$-coefficients, i.e.,

$$N\Delta_X = N(X\times\{x\}) + Z.$$

pf. (when X is rat'lly connected)

Claim: $\exists\ Y\times\mathbb{P}^1 \dashrightarrow X$ dominant, and $Y\times\{0\}\longrightarrow x$. $\dim Y = n-1$.

Y smooth proj. $\widetilde{Y\times\mathbb{P}^1}\xrightarrow{\varphi} X$ take $N=\deg\varphi$.

$$N\Delta_X = \varphi_*\Delta_{\widetilde{Y\times\mathbb{P}^1}} = \varphi_*(f^*\Delta_{Y\times\mathbb{P}^1} + \underset{E\times E}{\underset{\cap}{Z}})$$

$$= \varphi_*(f^*\Delta_Y\times(\{0\}\times\mathbb{P}^1+\mathbb{P}^1\times\{0\})) + Z'$$

$$= N(X\times\{x\}) + N(\{x\}\times X) + Z'$$

$$= N(X\times\{x\}) + Z''.$$

Defn. X has ~~rat'lly~~ universally trivial CH_0 if $CH_0(X_L)\xrightarrow{\sim}\mathbb{Z}$, $\forall\ L/K$ extn. and $\exists\ x\in X(K)$.

Prop. X has d.d. $/k$ iff X has universally trivial CH_0.

Cor. $/\mathbb{C}$, this is true for coh. d.d.

① X has coh. d.d. $\Leftrightarrow$ $X\times\mathbb{P}^r$ does.

Suppose $X\times Y$ has (coh). d.d. w/ pt $(x,y)\in X\times Y$

$(pr_{X\times X})_*[\Delta_{X\times Y}\cdot(X\times(y)\times X\times Y)] = \Delta_X$

$(pr_{X\times X})_*[(X\times Y\times\{x\}\times\{y\}+Z)\cdot X\times(y)\times X\times Y]$ gives d.d.

② birat'l invariance

$$\begin{tikzcd} & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[rr, dashed] & & Y \end{tikzcd}$$

can assume Z smooth.

$\pi_X^*(\Delta_X) = \Delta_Z + A$

$(\pi_Y)_*\Delta_Z = \Delta_Y$

countable closed subvarieties of d.d. locus.
Artin Mumford Invariants.

Prop. 1. $H^3_{tor}(V, \mathbb{Z})$ is stably bir'l invariant
pf. V_1, π_i blow-up smooth center Z_i
$V_1'' \dashrightarrow V_2$ by spectral sequence ... and H^1 has no torsion ...

Prop 2. coh. d.d. $\Rightarrow H^3_{tor}(X, \mathbb{Z}) = 0$.
pf. $\alpha = \pi_{2*}(\pi_1^* \alpha \cup \Delta) = \pi_{2*}(\pi_1^*\alpha \cup (x \times X + [Z]))$
$= \pi_{2*}(\pi_1^*\alpha \cup [Z])$
factors thru $H^1(\tilde{D}, Z)$ which has no torsion.

Prop 3. $X \xrightarrow{\text{flat}} \operatorname{Spec} R$, $Y \to X_0$ has trivial CH_0, and X_K has d.d.; then so does Y.

Prop 4. $X \xrightarrow{\text{flat}} B$ w/ B smooth, the locus where ~~X~~ fiber has d.d. is a countable union of closed subvarieties in B.

Stable Rationality & Grothendieck Ring.

k field, $V_k = \{$reduced, finite type schemes $/k\}$

Defn. $K_0(V_k) =$ Free abel. gp on $V_k / [X] = [Y] + [X - Y]$ $\forall Y \subseteq X$ closed subvariety. $[X]\cdot[Y] = [X \times Y]$ makes it a ring.

Rmk (1) k finite. Then $X \longrightarrow \# X$ gives $K_0(V_k) \longrightarrow \mathbb{Z}$
(2) $k = \mathbb{C}$. $X \longrightarrow \chi_{top}(X)$ gives $K_0(V_{\mathbb{C}}) \to \mathbb{Z}$.

Notation $\mathbb{L} = [\mathbb{A}^1]$

Thm (Larsen-Lunts) $K = \mathbb{C}$. Let X, Y be smooth proper vties$/\mathbb{C}$. Then $[X] = [Y]$ in $K_0/(\mathbb{L})$ iff X and Y are stably bir'l.

Let SB be the set of stable bir'l classes of var$/\mathbb{C}$.
$(X)\cdot(Y) = (X \times Y)$ makes SB commutative monoid.
$\mathbb{Z}[SB] =$ Free abel. gp on SB, is a commutative ring

Thm. $\exists\ K_0(V_{\mathbb{C}}) \xrightarrow{\varphi} \mathbb{Z}[SB]$, w/ kernel $= (\mathbb{L})$, sends $[X]$ to (X) for X smooth proper.

Prop. (0) $X = \coprod X_i$, $X_i \subset X$ locally closed, $[X] = \sum [X_i]$.
(1) $[\mathbb{P}^n] = \mathbb{L}^n + \cdots + 1$
(2) $E \to X$ vector bundle of rk n, $[E] = \mathbb{L}^n [X]$. $[\mathbb{P}E] = \mathbb{P}^{n-1}[X]$.
(3) $[X] = [X^{\text{non-sing}}] + [X^{\text{sing}}]$. together w/ resolution of singularity
$\Rightarrow K_0$ is generated by smooth ~~X~~ proper X.

(5) X, Y smooth $X \to Y$ blow up along smooth $Z \subset Y$ codim n.

$[X] = [Y - Z] + [Z] \times \mathbb{P}^{n-1}$.

$\varphi: Var/\mathbb{C} \longrightarrow \mathbb{Z}[SB]$ satisfying

A_n) For smooth complete X, $\varphi[X] = (X)$ $(\forall \dim X \leq n)$

B_n) For smooth compactification $X \hookrightarrow \bar{X}$ —"—

$\varphi[X] = \varphi(\bar{X}) - \varphi[\bar{X} - X]$

C_n) $\forall X$, $\varphi[X] = \varphi[X^{non\text{-}sing}] + \varphi[X^{sing}]$ —"—

D_n) $\forall X, Y$ $\dim \leq n$, $f: X \to Y$ s.t.
$\exists Y = \sqcup Y_i$ s.t. $f^{-1}(Y_i) \to Y_i$ is a proj bundle
then $\varphi[X] = \varphi[Y]$.

E_n) X of $\dim \leq n$, $Y \subset X$ closed

$\varphi[X] = \varphi[Y] + \varphi[X - Y]$.

F_n) For X, Y s.t. $\dim X + \dim Y \leq n$.

$\varphi[X \times Y] = \varphi[X] \cdot \varphi[Y]$.

Thm (Weak factorization) Let X, Y be smooth complete $/\mathbb{C}$ & $f: X \dashrightarrow Y$ birat'l. Then f factors blow up's and blow down's

$X \swarrow \searrow \swarrow \searrow \swarrow \searrow Y$ Moreover, if f is an isom on U, then centers can be chosen disjoint from U.

if $\alpha \in \ker(\varphi)$, $\alpha = \Sigma[X_i] - \Sigma[Y_j]$, X_i, Y_j smooth proper.

Then ~~#LHS~~ $\Sigma(X_i) = \Sigma(Y_j)$, so $\#LHS = \#RHS$. &

$(X_i) = (Y_j)$ in $SB \Rightarrow [X_i] = [Y_j] \mod \mathbb{L}$.

Family of Rationally Connected Varieties $/\mathbb{C}$.

Defn. X smooth, proj. is $\underline{RC}$ if for general $p, q \in X$ can be connected by chain of smooth rat'l curves.

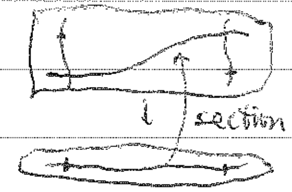
(Thm) Equivalently: $\forall p_1, \dots, p_n \in X$, $\xi_i \in T_{p_i}X$, $\exists$ a sm. rat'l curve passing (p_i, ξ_i) w/ ample normal bundle

Thm. X sm. proj., B sm. proj., $X \xrightarrow{\pi} B$ flat w/ general fiber R.C. then π has a section.

Rmk · change $\mathbb{C}$ to $k = \bar{k}$, it's still true (de Jong–Starr).

· $X/k(B)$ ~~geometrically~~ geom. $RC \Rightarrow X$ has $k(B)$ pts.

Cor. $X \xrightarrow{\pi} Y$, Y R.C., $\pi^{-1}(y)$ R.C. for general $y \Rightarrow X$ is RC.



pf of thm reduction to $B = \mathbb{P}^1$

$$\begin{array}{l} X \\ \downarrow \pi \\ B \xrightarrow{f} \mathbb{P}^1 \end{array}$$

construct $\tilde{X} \xrightarrow{\tilde{\pi}} \mathbb{P}^1$ s.t. $\tilde{\pi}^{-1}(p) = \prod_{q \in f^{-1}(p)} \pi^{-1}(q)$.

Assume $B = \mathbb{P}^1$, rel. $\dim(\pi) > 2$. Strategy: find $C \subseteq X$ general and deform and bend and break. Make $N_{C/X}$ more positive by attaching positive rat'l curves in fibers and smooth out.

Rat'l curves on C-Y 3-folds

Thm (Mori) bend & break X smooth, suppose $\exists C$ s.t. $K_X \cdot C < 0$, then $\exists$ rat'l curves R passing thru any pt in C, s.t. $0 < -K_X \cdot R \leq \dim X + 1$.

pf. $f : C \to X$. $F : C \times B \to X$

(1) $F(t, 0) = f(t) \quad \forall t \in C$. $\qquad \dim \operatorname{Def}(f, x) > \chi(f^* T_X)$

(2) $F(x, y) = f(x) \quad \forall y \in B$ $\qquad = -K_X \cdot f_* C - gn + n$.

(3) $\operatorname{Im} F$ is a surface $\qquad$ in char. p, precompose w/ Frob!

$$\begin{tikzcd} C \times \bar{B} \arrow[rr, dashed] & & X \\ & X' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

in char. 0, do reduction

Thm (Kollar-Miyaoka-Mori) X sm., $-K_X$ ample, thru any two pts $\exists$ a rat'l curve C, s.t. $0 < -K_X \cdot C \leq M_n$ ($\sim (n+1)n$)

Cor Smooth Fano varieties are bounded family.

pf. $D = -K_X$. $\quad 0 \to mD \otimes m_x^{\alpha} \to mD \to mD \otimes \mathcal{O}_X / m_x^{\alpha} \to 0$

$h^0 \sim m^n D^n \quad h^0 \sim \alpha^n$.

$\alpha = \beta m \qquad D^n - \varepsilon \leq \beta^n \leq D^n$

$\exists D_x \sim mD$, $\operatorname{mult}_x D_x \geq \beta \cdot m$ $\quad$ let $y \notin \operatorname{Supp} D_x$.

$\beta m \leq D_x \cdot C = -m K_X \cdot C \leq m M_n \Rightarrow (-K_X)^n \leq M_n^n$.

Thm (Mori) X sm., there are $\leq$ countable many rat'l curves, s.t.

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \cdot (\cdot) \geq 0} + \sum_i \mathbb{R}_{\geq 0} [C_i].$$

Thm (Mukai-Mori, Bogomolov, Mumford) Any K3 has a rat'l curve.

pf. (X, H) X K3, H ample $H^2 = 2g-2$, $\operatorname{Def}(X, H)$ is smooth. In this family, for Kummer K3's, we can exhibit rat'l curves, then try to deform.

Thm (X, Δ) KLT, then $\exists \leq$ countably many rat'l curves C_i, s.t.

$$\overline{NE}(X) = \overline{NE}(X)_{K_X + \Delta \geq 0} + \sum_i \mathbb{R}_{\geq 0} [C_i].$$

$$0 < -(K_X + \Delta) \cdot C_i \leq 2n + 1.$$

Cor. X is CY, suppose $\exists$ an effective divisor which is not nef. Then $\exists$ a rat'l curve.

pf. $(X, \varepsilon E)$ $\quad K_X + \varepsilon E$ not nef.

Thm (Wilson) X CY 3fold, w/ $\rho(X) \geq 19$. Then $\exists$ a rat'l curve.

π_1 of unirat'l vties.

$V/\mathbb{C}$, smooth, proj., unirat'l.

$\exists\, f : \mathbb{P}^n \dashrightarrow V$ rat'l dominant.

Main Thm: $\pi_1(V) = 0$ for V as above.

Step 1: $\chi(V, \mathcal{O}_V) = 1$.

as for V above, we have

① $h^{0,q} = h^{q,0} = 0 \quad \forall q > 0$

② hence $\chi(V, \mathcal{O}_V) = 1$.

Step 2. $\pi_1(V) < \infty$

$$\begin{tikzcd} & V' \arrow[d, "\pi"] \\ \mathbb{P}^n - W \arrow[ur, "g"] \arrow[r, "f"] & V \end{tikzcd}$$

π universal covering

look at $U \subseteq V$ where f is a covering, we see

$\deg(f) = \deg(g) \cdot \deg(\pi)$, so $\deg(\pi) < +\infty$.

Lemma In the situation above, V' is proj., nonsingular, alg. vty.

Lemma X, Y, Z alg. f, π morphism$/\mathbb{C}$, $\pi^{-1}(x) < \infty$, $\forall x \in X$.

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "\pi'"'] & Y \arrow[d, "\pi"] \\ Z \arrow[u, bend right, "g'"'] \arrow[ur, "g"] \arrow[r, "f"] & f \end{tikzcd}$$

then g is algebraic.

Step 3. Conclusion!

V' is also unirat'l, so $\chi(V', \mathcal{O}_{V'}) = 1$.

Hence $\pi_1(V) = 1$.

Rmk this fails in char p.

Shioda's example: $V' : X_0^5 + \cdots + X_3^5 = 0 \subseteq \mathbb{P}^3$ is unirat'l

$\mathbb{Z}/5 \circlearrowright V'$. $[x_i] \longmapsto [\zeta_5^i x_i]$.

$V = V'/\mathbb{Z}_5$ is unirat'l, $\chi(V', \mathcal{O}_{V'}) = 5$

$\chi(V, \mathcal{O}_V) = 1$.

Hassett - Pirutka - Tschikel

Thm. consider smooth hypersurface $X \subseteq \mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(2,2)/_{k=\mathbb{C}}$.
Then ① a very general one is not stably rat'l.
② the collection of rat'l ones is Euclidean dense in moduli.

Idea of ②: $X \xrightarrow{\pi} \mathbb{P}^2$ general fiber X_η is a quadric surface.

X is rat'l $\Leftarrow X_\eta$ is rat'l $/K = k(\mathbb{P}^2)$

$\Leftarrow X(K) \neq \phi \Leftarrow Q(K') \neq \phi$ for $[K':K]$ odd.

$\Leftarrow \exists\, \alpha \in CH_2(X)$ s.t. $\alpha \cdot \pi^{-1}(pt)$ odd

$\Leftarrow \exists\, \gamma \in H^4(X,\mathbb{Z})$ s.t. $\gamma \cdot [\pi^{-1}(pt)]$ odd

Hassett's thesis or integral Hodge conj.

Voisin's trick: $H^4_{van}(X) = H^4(X,\mathbb{Z}) / H^4(\mathbb{P}^3 \times \mathbb{P}^2, \mathbb{Z})$ has only
$H^{3,1} \oplus H^{2,2} \oplus H^{1,3}$, If $H^{3,1}$ moves enough in the family of
dim'n 3 27 3 X's, then every γ in some open
$U \subseteq H^4_{van}(X,\mathbb{Q})$ occurs as a Hodge cycle on a nearby ~~cycle~~ fiber.

Quadrics in $\mathbb{P}^3$: $G = \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1)) \hookrightarrow GL_4$.

$1 \longrightarrow G^\circ \longrightarrow G \longrightarrow \{1, \text{flip}\} = \{\pm 1\} \longrightarrow 0$.

$1 \to \mathbb{G}_m \longrightarrow GL_2 \times GL_2 \longrightarrow G^\circ \longrightarrow 1$

$H^1(K,G) = \{\text{smooth quadric surfaces } Q \text{ in } \mathbb{P}^3_K\}/\cong$ $\quad Q: \{g=0\}$

$\downarrow \qquad\qquad \downarrow$

$H^1(K,\{\pm1\}) = K^\times / K^{\times 2}$ $\qquad$ $\mathrm{disc}(g)$.

Lemma 1. If $\mathrm{disc}(g)$ not square, then $H^2(K,\mu_2) \subseteq Br(K)[2] \hookrightarrow Br(K(Q))$

Lemma 2: If $Q: X_0^2 - aX_1^2 - bX_2^2 + abX_3^2 = 0$, then $(a,b)_2 = $ Br. class of
$K\langle u,v\rangle / (u^2-a, v^2-b, uv+vu)$ maps to 0 in $Br(Q)$.

pf of ① find a singular $X \subseteq \mathbb{P}^3 \times \mathbb{P}^3$ s.t. $H_{nr}(k(X)/k, \mu_2) \neq 0$ & $\exists$ resolution
$\tilde{X} \to X$ which is univ. CH_0 trivial.

Defn. If K/k (f.g.) field extn $H^2_{nr}(K/k, \mu_n) = \{\alpha \in H^2(K,\mu_n)$
s.t. for any discrete valuation v on $k(X)$ trivial on k, the
image $\partial_v(\alpha) \in H^1(\kappa(v), \mu_n)$ is zero$\}$.

Lemma. $H^2_{nr}(k(X)/k, \mu_n)$ is stably bir'l inv.

X: $yz \cdot s^2 + xz \cdot t^2 + xy \cdot u^2 + F(x,y,z) v^2 = 0$.

$F(x,y,z) = x^2 + y^2 + z^2 - 2(xy + yz + xz)$